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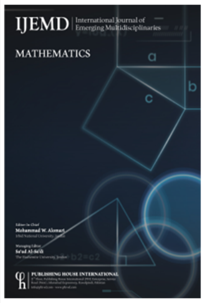

Numerical Solution of an Inviscid Burger Equation with Cauchy Conditions

Muhammad Zahid

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Abstract

This article deals with the solution of Cauchy problem for Inviscid Burger equation. Various numerical techniques like, Upwind non Conservative, Upwind Conservative, Lax Friedrich, Lax Wendorff, Mac Cormack, are used to solve initial-value problems for the Inviscid Burger equation. Through various model problems the efficiency and accuracy of the techniques has been shown via graphical and tabulated form with the exact solution.

Keywords: Burger's equation; Upwind conservative and non-conservative method; Upwind conservative method; Lax Friedrich method; Lax Wendroff method; Mac Cormack method, Initial-value problem.

1. Introduction

Bateman [1] derived Burger's Equation in physical context. Afterward Fay [2] gave a solution of Burger's equation in series form. Burger [3] emphasized the importance of this equation and gave a special solution. Hopf and Cole [4-5] transformed the Burger equation in to linear Heat equation and called as Cole-Hopf transformation. Cole discovered the Fay series and gave the approximate solution of Burger equation for sinusoidal Cauchy condition. For the study of the propagation of one dimensional acoustic signals of finite amplitude, D.T Black stock and M.J.Lighthill [6-7] used Burger's equation. For the discussion of shock structure in Navier-Stokes fluids, W.D. Hayes [8] used Burger equation. Benton [9-10] gave it's an exact solution. Riccati solution of Burger's equation was obtained by Rodin[11] without using given conditions. 35 distinct solutions of Burger equation with Cauchy conditions in the infinite domain and two solutions of IBV problems was published by Benton and Platzman [12]. Ames [13] applied Morgan-Michal method for finding the proper groups for Burger's equation without any auxiliary conditions. Finite element method was applied by Varoglu and Finn [14] for solving the Burger equation taking the space time elements and characteristic.

Weiss et al. [15] defined the painleve property of the PDE and found the integrability, the Backlund transforms, and the Lax pairs of the Burger's equation. Vorus [16] studied the perturbed vortex sheet and found Burger's equation on a moving medium. He integrated the Burger equation and its condition to avoid the non linearity and

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solve by using Cole- Hopf transformation. The Cole-Hopf transformation was used by Shtelen [17]. In the same year, Fushchich et al. [18] discussed Burger's equation and showed that it is invariant with respect to space and time translations. An integral term is added to study the stability of solution of controlled Burger equation by Peralta-Fabi and Plaschko [19]. Ozis and Ozdes [20] used the variational method to find an approximate solution in the form of a series. Mazzia and Mazzia [21] converted Burger's equation into a system of ODEs, and used the transverse scheme to solve it. Miroslav Krstic [22] used Burger equation with Neumann and Drichtle boundary conditions to find the non-linear boundary control laws to get global asymptotic stability. He also studied the stochastic Burger's Equation. L. Frachebourg, Ph.A.Martin, J.Piasecki proved that limiting distributions are same as statistics of shocks in Burger's turbulence. Adel N.Boules [23] presented a finite element approximation of the two dimensional steady Burgers 'equation and a conjugate gradient approach is used to solve the resulting finite element equations. G.W.Wei,Yun Gu proposed a novel scheme for solving Burger equation with all possible values of Reynolds numbers.

Kamran Mohseni, Hongwu Zhao, Jerrold E.Marsden [24]solved Burger equation numerically to investigate the regularization effect and they showed that the thickness of regularized shock is controlled by width of filter. John P.Boyd [25] used Burgers' equation to investigate the post breaking behaviour of fronts. Kanti Pandey, Lajja Verma [26] generate the numerical solutions of the Burger's equation by applying Crank-Nicolson method directly to the Burger's equation i.e. did not use Hopf- Cole transformation to reduce Burger's equation in to the linear Heat equation.Kemal Altiparmak, Turgut Ozis [27] solved one dimensional Burger equation numerically by using Pade approximation with factorization scheme and results so obtained are compared with exact solutions. Rajan Arora, Md.junaid Siddiqui and V.P Singh used Reduced Differential Transform method to find the exact analytical solution of inviscid Burger equation. E.J.Kansa, Jurgen Geiser [28] used domain decomposition method on the 4D Burger equation. Mohammadreza Askaripour Lahiji,Zainal Abdul Aziz, Mahdi Ghanbari and Hassan Panj Mini [29] used the exponential time differencing RK-4 method (ETDRK4) to solve the diagonal example of a well known non linear partial differential equation (PDE) in the form of Burgers equation and used Fourier transformation to solve the equation. Alicia Cordero, Antonio, Franques and Juan R.Torregrosa [30] used the Crank-Nicholson method to solve the Burger equation and used different scheme to solve non linear system without using Hopf-Cole transformation. Saida Bendaas [31]used the non standard analysis for both the viscid and inviscid Burger's equation with Cauchy conditions in the half plane $x \in R, T > 0$.

Here we used various numerical techniques like, Upwind non Conservative, Upwind Conservative, Lax Friedrich, Lax Wendroff, and Mac Cormack to solve initial-value problems for the Inviscid Burger equation. We also used the same techniques for the solution of initial-boundary value problems for the Inviscid Burger equation.

2. Analysis of Finite Difference Schemes

Here, we present brief introduction of five numerical schemes for solving initial value problems and initial boundary value problems for Inviscid Burger equation.

2.1 Up-wind non Conservative Scheme

Consider the Inviscid Burger equation in the Quasi-linear form

$$u_t + uu_x = 0 \tag{1}$$

using forward in time and backward in space, we get finite difference scheme as

$$U_j^{n+1} = U_j^n - \frac{k}{h} U_j^n \left(U_j^n - U_{j-1}^n\right), \tag{2}$$

which is called up-wind non conservative scheme.

2.2 Up-wind Conservative Method

Since inviscid Burger equation can be written in the conservation form i.e.

$$u_t + [f(u)]_x = 0, \tag{3}$$

where

$$f(u) = \frac{1}{2}u^2,$$

if u_t is replaced by forward difference approximation and the $[f(u)]_x$ is replaced by backward in space we get

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[f(U_j^n) - f(U_{j-1}^n)\right], \tag{4}$$

which is required upwind conservative scheme. This can also be written as

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left[\frac{1}{2}(U_j^n)^2 - \frac{1}{2}(U_{j-1}^n)^2\right] \tag{5}$$

2.3 Lax-Friedrich's Method

The linear convection equation is given as

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \tag{6}$$

using forward in time and central in space, we get the numerical scheme for the equation (6) as

$$u_j^{n+1} = su_{j+1}^n + (1 - 2s)u_j^n + su_{j-1}^n, \tag{7}$$

where $S = c\,\frac{\Delta t}{\Delta x}$ is the CFL number. If we replace u_j^n by the averaged term $\frac{1}{2}(u_{j+1}^n + u_{j-1}^n)$ we get new scheme which is called Lax Method. So equation (7) becomes

$$u_j^{n+1} = \frac{1}{2}(u_{j+1}^n + u_{j-1}^n) - \frac{1}{2}c\,\frac{\Delta t}{\Delta x}(u_{j+1}^n - u_{j-1}^n) \tag{8}$$

2.4 Lax-Wendroff Scheme

The Lax-Wendroff method is one of the first second -order finite difference method for hyperbolic PDE. The Lax -Wendroff scheme for above Inviscid Burger Equation is

$$U_j^{n+1} = U_j^n - \frac{k}{h}\frac{f(U_{j+1}^n) - f(U_{j-1}^n)}{2} + \frac{1}{2}\left(\frac{k}{h}\right)^2\left[A_{j+\frac{1}{2}}^n\left(f(U_{j+1}^n) - f(U_{j-1}^n)\right) - A_{j-\frac{1}{2}}^n\left(f(U_j^n) - f(U_{j-1}^n)\right)\right], \tag{9}$$

where $A_{j\pm\frac{1}{2}}$ is the jacobian matrix $A(u) = f'(u)$ evaluated at half interval i.e $A_{j+\frac{1}{2}} = \frac{U_j^n+U_{j+1}^n}{2}$ and $A_{j-\frac{1}{2}} = \frac{U_j^n+U_{j-1}^n}{2}$.For Burger equation we have $f'(u) = u$ so

$$U_j^{n+1} = U_j^n - \frac{k}{2h}\left[\frac{1}{2}\left(U_{j+1}^n\right)^2 - \frac{1}{2}\left(U_{j-1}^n\right)^2\right] + \frac{1}{2}\left(\frac{k}{h}\right)^2\left[\left(\frac{U_j^n + U_{j+1}^n}{2}\right)\left(\frac{1}{2}\left(U_{j+1}^n\right)^2 - \frac{1}{2}\left(U_j^n\right)^2\right) - \left(\frac{U_j^n + U_{j-1}^n}{2}\right)\left(\frac{1}{2}\left(U_j^n\right)^2 - \frac{1}{2}\left(U_{j-1}^n\right)^2\right)\right], \quad (10)$$

is Lax-Wendroff scheme.

2.5 McCormack Scheme

Another method of the same type is known as Mac Cormack's method. It is predictor- corrector method. It is much easier method to apply than the Lax- Wendroff scheme because the Jacobian does not appear. In this method first uses the forward difference (FD) and then backward difference (BD) to achieve second order accuracy

$$u_j^{\overline{n+1}} = u_j^n - \frac{k}{h}\left[f\left(U_{j+1}^n\right) - f\left(U_j^n\right)\right] \quad (11)$$

$$u_j^{n+1} = \frac{1}{2}\left[\left(u_j^n + u_j^{\overline{n+1}}\right) - \frac{k}{h}\left(f\left(u_j^{\overline{n+1}}\right) - f\left(u_{j-1}^{\overline{n+1}}\right)\right)\right]. \quad (12)$$

3. Implementation of Numerical Techniques

The numerical schemes discussed above section are implemented on initial value problems with different Cauchy conditions. The detail is given as below.

3.1.1 Model Problem-A[32]

Consider the initial value problem for the Inviscid Burger equation

$$u_t + uu_x = 0, \quad (13)$$

subject to the Cauchy condition

$$u(x,0) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0 \end{cases} \quad (14)$$

Its exact solution is given by

$$u(x,t) = \begin{cases} 1 & x - st < 0 \\ 0 & x - st > 0 \end{cases}, \text{where } s = \frac{1}{2}. \quad (15)$$

After applying finite difference schemes (2),(5),(8),(10), and (12) by using MATLAB, the graphical comparison is given as under

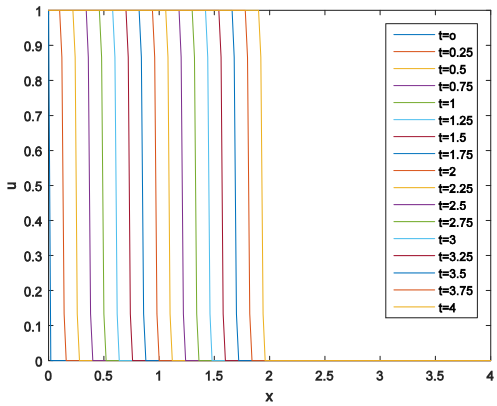

Fig.1(a). Upwind non Conservative.

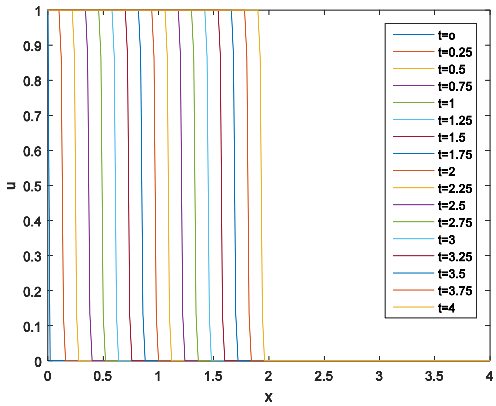

Fig.1(b). Upwind Conservative.

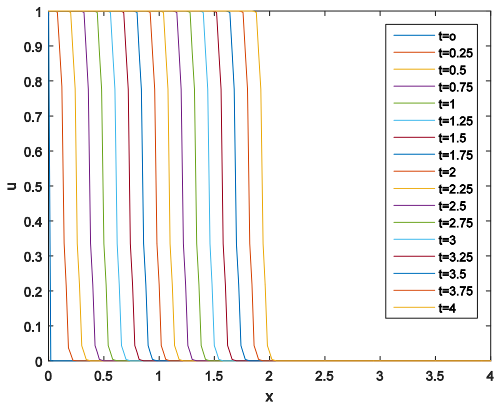

Fig.1(c). Lax Friedrich

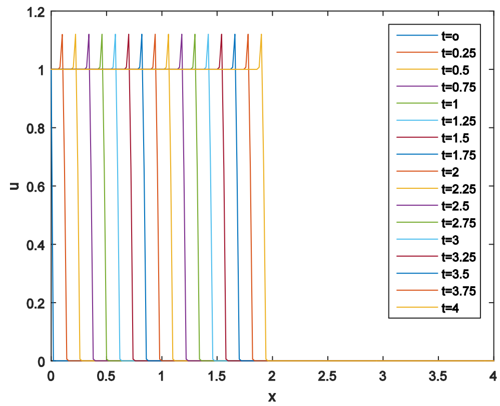

Fig.1(d). Lax Wendroff

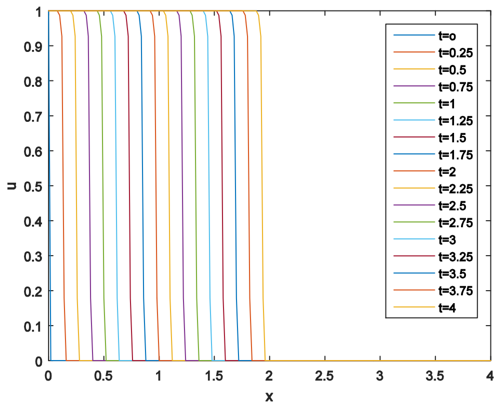

Fig.1(e). McCormack.

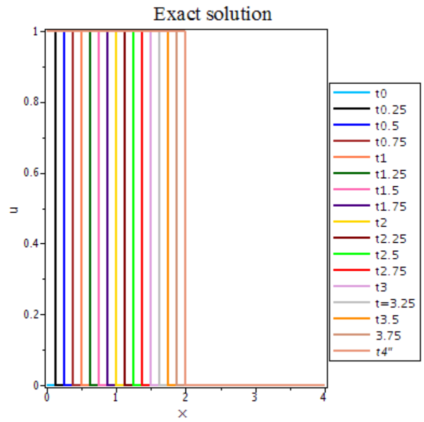

Fig.1(f). Exact solution.

Table 1. Comparison of numerical and exact solution for different values of x&$t = 4$.

i	x	Upwind Non Conservative	Upwind Conservative	Lax Friedrich	Lax Wendroff	McCormack	Exact solution
1	0	1	1	1	1	1	1
2	0.02	1	1	1	1	1	1
3	0.04	1	1	1	1	1	1
4	0.06	1	1	1	1	1	1
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
196	3.92	0	0	$5.05e(-56)$	0	0	0
197	3.94	0	0	$9.56e(-58)$	0	0	0
198	3.96	0	0	$4.80e(-58)$	0	0	0
199	3.98	0	0	$4.56e(-60)$	0	0	0
200	4.00	0	0	0	0	0	0

3.1.2 **Model Problem-B**[33]-[34]

Consider the initial value problem for the inviscid Burger equation

$$u_t + uu_x = 0, \tag{16}$$

subject to the Cauchy condition

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{17}$$

with exact solution is given by

$$u(x,t) = \begin{cases} 0 & if\ \frac{x}{t} < 0 \\ \frac{x}{t} & 0 < \frac{x}{t} < 1 \\ 1 & \frac{x}{t} > 1 \end{cases}, \tag{18}$$

After applying finite difference schemes (2),(5),(8),(10), and (12) and using MATLAB, the graphs and comparison is as under

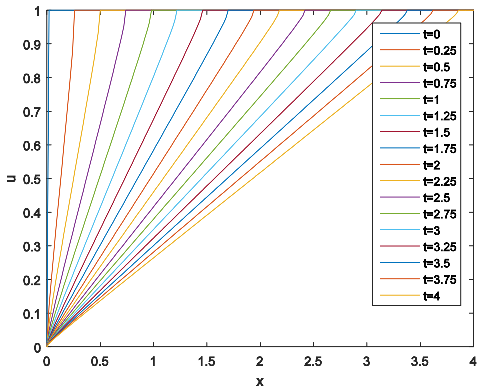

Fig.2(a). Upwind Non Conservative

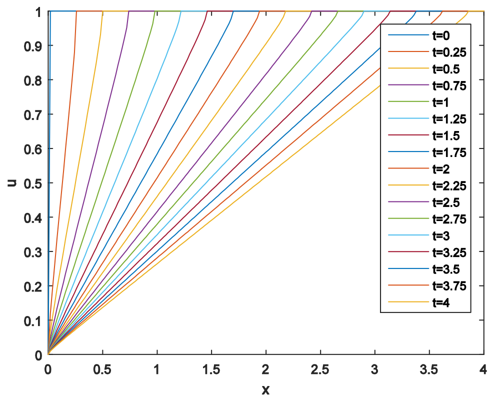

Fig.2(b). Upwind Conservative.

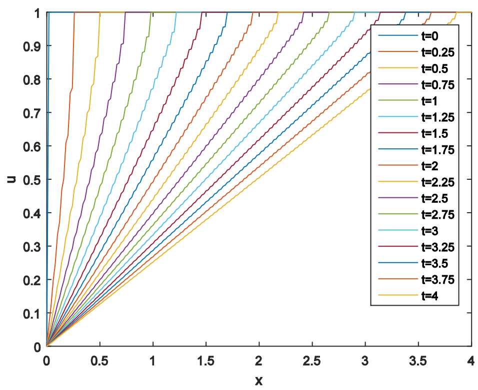

Fig.2(c). Lax Friedrich.

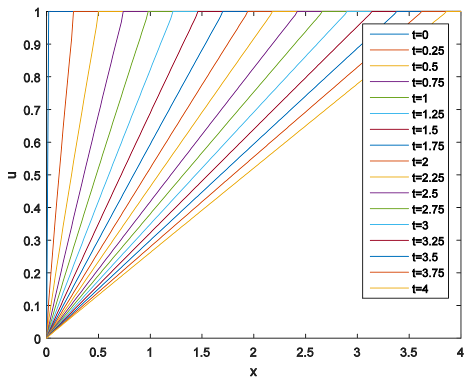

Fig.2(d). Lax Wendroff.

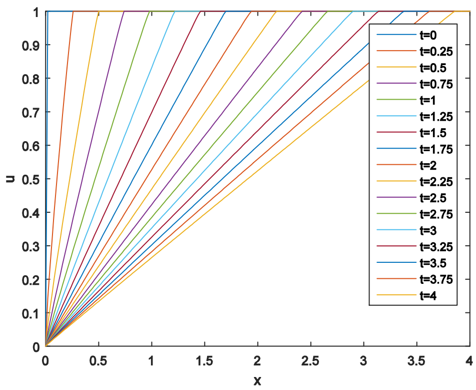

Fig. 2(e). McCormack.

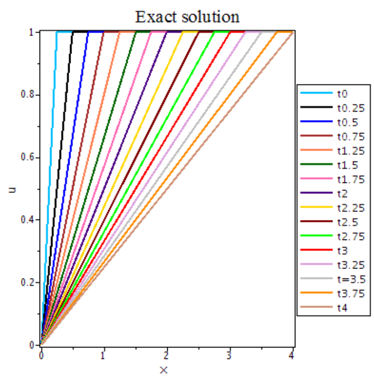

Fig.2(e). Exact solution.

Table 2. Comparison of numerical and exact solution for different values of $x\&t = 4$.

i	x	Upwind Non Conservative	Upwind Conservative	Lax Friedrich	Lax Wendroff	McCormack	Exact solution
1	0	0	0	0	0	0	0
2	0.02	0.0096	0.0096	0.0048	0.0059	0.0061	0.0050
3	0.04	0.0156	0.0156	0.0096	0.0111	0.0114	0.0100
4	0.06	0.0211	0.0211	0.0145	0.0162	0.0166	0.0150
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
196	3.92	0.9681	0.9681	0.9578	0.9739	0.9770	0.9800
197	3.94	0.9734	0.9734	0.9687	0.9788	0.9819	0.9850
198	3.96	0.9788	0.9788	0.9689	0.9837	0.9867	0.9900
199	3.98	0.9843	0.9843	0.9806	0.9886	0.9915	0.9950
200	4.00	0.9903	0.9903	1	1	1	1

3.1.3 **Model Problem-C**[32]

Consider the initial value problem for the inviscid Burger equation

$$u_t + uu_x = 0, \tag{19}$$

subject to the Cauchy condition

$$u(x,0) = \begin{cases} 1, & x < 0 \\ 1 - x, & 0 \le x \le 1. \\ 0, & x > 1 \end{cases} \tag{20}$$

and the exact solution is given by

$$u(x,t) = \begin{cases} 1 & x < t \\ \dfrac{1-x}{1-t} & t \le x \le 1. \\ 0 & x > 1 \end{cases} \tag{21}$$

After applying finite difference schemes (2),(5),(8),(10), and (12) and using MATLAB ,the graphs and comparison is as under

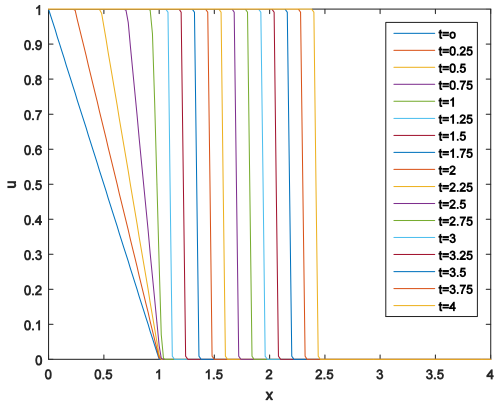

Fig.3(a). Upwind Non Conservative.

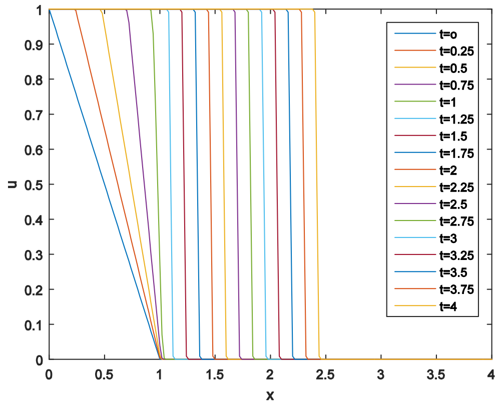

Fig.3(b). Upwind Conservative.

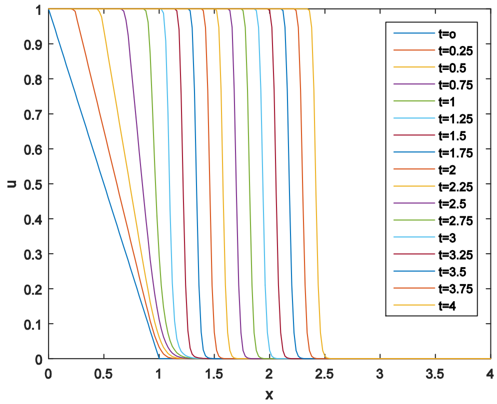

Fig.3(c). Lax Friedrich.

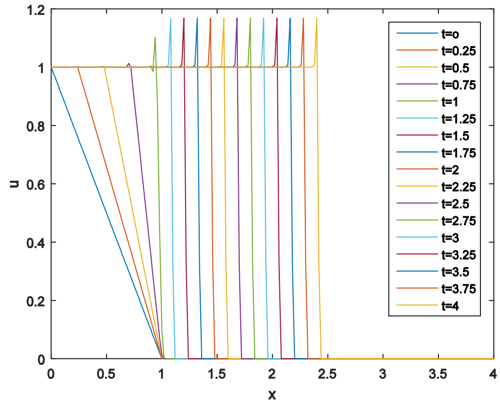

Fig.3(d). Lax Wendroff.

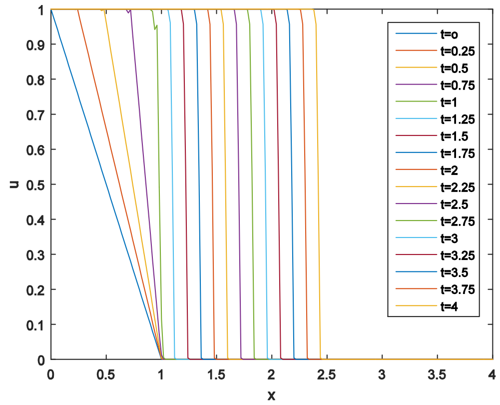

Fig.3(e). McCormack.

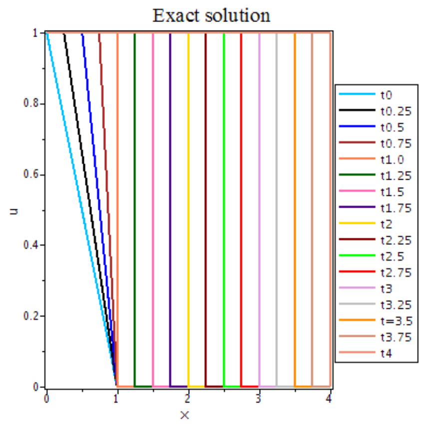

Fig.3(f). Exact solution.

Table 3. Comparison of numerical and exact solution for different values of x&$t = 4$.

i	x	Upwind Non Conservative	Upwind Conservative	Lax Friedrich	Lax Wendroff	McCormack	Exact solution
1	0	1	1	1	1	1	1
2	0.02	1	1	1	1	1	1
3	0.04	1	1	1	1	1	1
4	0.06	1	1	1	1	1	1
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
196	3.92	0	0	0		0	1
197	3.94	0	0	0	$6.3968e-30$	0	1
198	3.96	0	0	0	$1.6374e-30$	0	1
199	3.98	0	0	0	$8.0141e-31$	0	1
200	4.00	0	0	0	0	0	0

3.1.4 Model Problem-D[31],[34]

Consider the initial value problem for the inviscid Burger equation

$$u_t + uu_x = 0, \tag{22}$$

subject to Cauchy condition

$$u(x,0) = \begin{cases} 0.8 & x < 0 \\ 0.2 & x \geq 0 \end{cases} \tag{23}$$

and exact solution is given by

$$u(x,t) = \begin{cases} 0.8 & x - st < 0 \\ 0.2 & x - st > 0 \end{cases}, \text{where } s = 0.5(0.8 + 0.2). \tag{24}$$

After applying finite difference schemes (2),(5),(8),(10), and (12) and using MATLAB, the graphs and comparison is as under

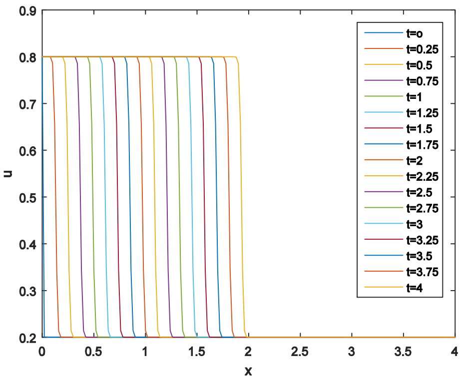

Fig.4(a). Upwind Non Conservative

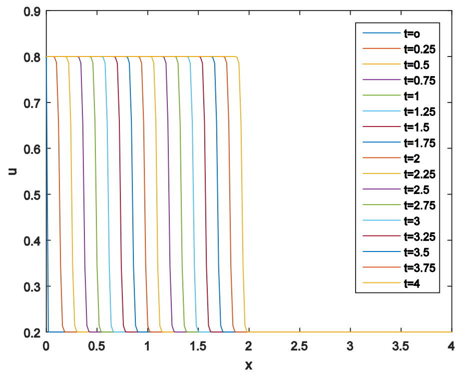

Fig.4(b). Upwind Conservative

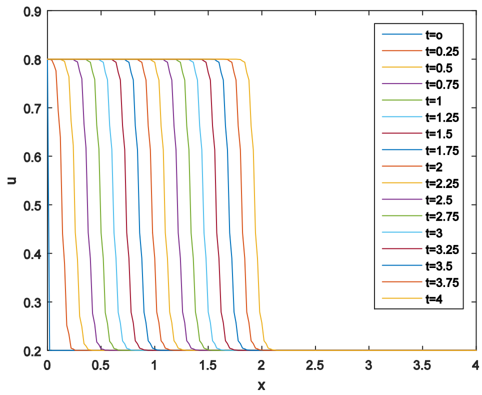

Fig.4(c). Lax Friedrich.

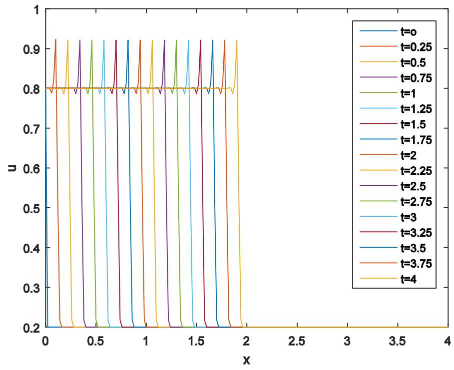

Fig. 4(d). Lax Wendroff.

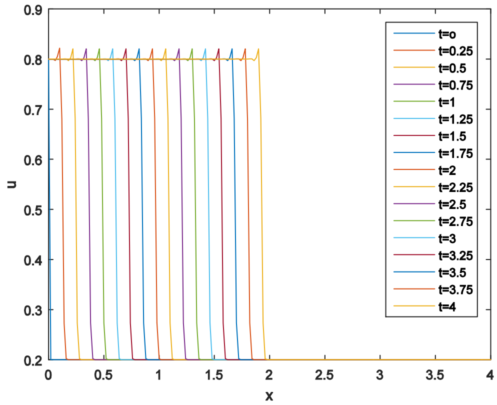

Fig.4 (e). McCormack.

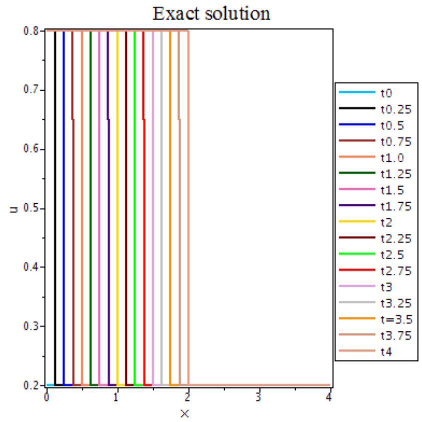

Fig. 4(f). Exact Solution.

Table 4. Comparison of numerical and exact solution for different values of x&$t = 4$.

i	x	Upwind Non Conservative	Upwind Conservative	Lax Friedrich	Lax Wendroff	McCormack	Exact solution
1	0	0.8	0.8	0.8	0.8	0.8	0.8
2	0.02	0.8	0.8	0.8	0.8	0.8	0.8
3	0.04	0.8	0.8	0.8	0.8	0.8	0.8
4	0.06	0.8	0.8	0.8	0.8	0.8	0.8
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
:	:	:	:	:	:	:	:
196	3.92	0.2	0.2	0.2	0.2	0.2	0.2
197	3.94	0.2	0.2	0.2	0.2	0.2	0.2
198	3.96	0.2	0.2	0.2	0.2	0.2	0.2
199	3.98	0.2	0.2	0.2	0.2	0.2	0.2
200	4.00	0.2	0.2	0.2	0.2	0.2	0.2

3. Conclusion

The numerical techniques upwind Non Conservative, Upwind Conservative, Lax Friedrich, Lax Wendroff, McCormack have been used to find the solutions of the Inviscid Burger's equation with initial conditions. The efficiency of the techniques was tested on the equation with different initial conditions. In IVP, the comparison of the numerical solution obtained by these techniques to the exact solution is shown by mean of graphs and tables which are in good agreement with the exact solutions.

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