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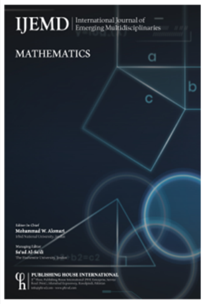

Theory and Computations for the System of Integral Equations via the use of Optimal Auxiliary Function Method

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Abstract

This paper extends the application of Optimal Auxiliary Function Method (OAFM) to the system of integral equations. The system of Volterra integral equations of second kind are taken as test examples. The results obtained by proposed method are compared with different methods i.e., Biorthogonal systems in a Banach space Fixed point, the implicit Trapezoidal rule in conjunction with Newton's method, and Relaxed Monte Carlo method (RMCM). The results revealed that OAFM is more efficient, simple to apply, and fast convergent. The auxiliary functions used in the method control its convergence. The values of arbitrary constants involved in the auxiliary functions are calculated optimally using the method of least square.

Keywords: Systems of nonlinear Volterra integral equations of the second kind; Least square method; Optimal auxiliary function method; auxiliary functions.

1. Introduction

Integral equation has been one of the principal tools in various areas of applied mathematics, physics, and engineering, hence, the literature on integral equations and their applications is vast. The principal investigators of the theory of integral equations are Vito Volterra and Ivar Fredholm, together with David Hilbert and Erhard Schmidt. Solutions of integral equations play an important role in the proper understanding of qualitative features of many phenomena and processes in various areas of natural science. Lots of equations of physics, chemistry, and biology contain functions or parameters which are obtained from experiments and hence are not strictly fixed. Therefore, it is expedient to choose the structure of these functions so that it would be easier to analyze and solve the equation. As a possible selection criterion, one may adopt the requirement that the model integral equation admits a solution in a closed form. Integral equations are used in many ways [1-7]. Some of the numerical and analytical methods for solving linear and nonlinear integral equations are, Collocation Method [8], Homotopy

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Perturbation Method (HPM) [9], Homotopy Analysis Method (HAM) [10] and the Optimal Homotopy Asymptotic Method (OHAM) [11].

Above mention methods have their own set of benefits and drawbacks. Marinca et al. [12-13] presented an Optimal Auxiliary Function method, for the solution of differential equation. They used to find the series solution of the thin-film of a fourth-grade fluid down vertical cylinder and permanent Magnet synchronous generator [14-15]. Later on, the method was extended by many authors for solution of fractional order high dimensional equations, partial differential equation and integro differential equations [16-18]. Our aim in this paper is to extend the application of OAFM to system of integral equations. Different types of system of Volterra integral equations are solved using the proposed method. Comparison with methods i.e., Biorthogonal systems in a Banach space Fixed point, the implicit Trapezoidal rule in conjunction with Newton's method, Relaxed Monte Carlo method revealed that proposed method is very reliable and effective for solving system of integral equations. The accuracy of proposed method can further be improved by increasing the auxiliary constants in auxiliary functions.

The organized part of paper is as follows; first section was the introduction. The second section is about OAFM. Third section, is taken for the implementation of OAFM with examples. While the last section is devoted for conclusion.

2. Basic idea of OAFM for system of integral equations

Step 1: In general, we consider system of integral equations,

$$\begin{aligned}\phi(\tau) &= x_1(\tau)+\lambda_1\int_0^1\left(\aleph_1(\tau,\eta)\phi(\eta)+\tilde{\aleph}_1(\tau,\eta)\varphi(\eta)\right)d\eta,\\ \varphi(\tau) &= x_2(\tau)+\lambda_2\int_0^1\left(\aleph_2(\tau,\eta)\phi(\eta)+\tilde{\aleph}_2(\tau,\eta)\varphi(\eta)\right)d\eta.\end{aligned} \tag{4}$$

Step 2: To determine the approximate solution for Eq. (4), here we consider two components of series solution,

$$\begin{aligned}\tilde{\phi}(\tau) &= \phi_0(\tau)+\phi_1\left(\tau,a_{1i},a_{1j}\right), i=1,2,3,...m,\ j=m+1,m+2,...n,\\ \tilde{\varphi}(\tau) &= \varphi_0(\tau)+\varphi_1\left(\tau,a_{2k},a_{2l}\right), k=1,2,3,...p, l=p+1,p+2,...q.\end{aligned} \tag{5}$$

Step 3: By equating source part and nonlinear part, we get the following.

$$\begin{aligned}L_1(\phi(\tau)) &= x_1(\tau),\\ L_2(\varphi(\tau)) &= x_2(\tau),\\ N_1(\phi(\tau)) &= \lambda_1\int_g^h\left(\aleph_1(\tau,\eta)\phi(\eta)+\tilde{\aleph}_1(\tau,\eta)\varphi(\eta)\right)d\eta,\\ N_2(\varphi(\tau)) &= \lambda_2\int_g^h\left(\aleph_2(\tau,\eta)\phi(\eta)+\tilde{\aleph}_2(\tau,\eta)\varphi(\eta)\right)d\eta.\end{aligned} \tag{6}$$

Step 4: Zero order $\phi_0(\tau),\varphi_0(\tau)$ approximation can be obtained from the following equations,

$$\begin{aligned}\phi_0(\tau) &= -x_1(\tau),\\ \varphi_0(\tau) &= -x_2(\tau).\end{aligned} \tag{7}$$

Step 5: Non-linear part can be obtained from the following equations,

$$\begin{aligned}N_1(\phi(\tau)) &= \lambda_1\int_g^h\left(\aleph_1(\tau,\eta)\phi(\eta)+\tilde{\aleph}_1(\tau,\eta)\varphi(\eta)\right)d\eta,\\ N_2(\varphi(\tau)) &= \lambda_2\int_g^h\left(\aleph_2(\tau,\eta)\phi(\eta)+\tilde{\aleph}_2(\tau,\eta)\varphi(\eta)\right)d\eta.\end{aligned} \tag{8}$$

Step 6: The first order solution $\phi_1(\tau),\varphi_1(\tau)$ can be obtained from the following equations,

$$\begin{aligned}
&\phi_1(\tau,\mathrm{a}_{1i},\mathrm{a}_{1j}) = -F_1(\phi_0(\tau),\mathrm{a}_{1i})\int_g^h \left(\aleph_1(\tau,\eta)\phi(\eta)+\tilde{\aleph}_1(\tau,\eta)\varphi(\eta)\right)d\eta \\
&-F_2(\phi_0(\tau),\mathrm{a}_{1j}), \\
&\varphi_1(\tau,\mathrm{a}_{2k},a_{2l}) = -F_3(\varphi_0(\tau),\mathrm{a}_{2k})\int_g^h \left(\aleph_2(\tau,\eta)\phi(\eta)+\tilde{\aleph}_2(\tau,\eta)\varphi(\eta)\right)d\eta \\
&-F_4(\varphi_0(\tau),\mathrm{a}_{2l}).
\end{aligned} \tag{9}$$

Where $i=1,2,3,\ldots m,\ j=m+1,m+2,\ldots n,\ k=1,2,3,\ldots p,\ l=p+1,\ p+2,\ldots q.$
Substitute Eq. (7) and Eq. (9) into Eq. (5) we get our approximate solution.

Remark 1: F_1 and F_2 are auxiliary functions whose values are determined by initial approximation $\phi_0(\tau),\varphi_0(\tau)$ and a number of unknown constants a_{1i} and a_{1j}.

Remark 2: F_1 and F_2 are not unique, and they are in the same form as source part and non-linear part or the combination of both.

Remark 3:
1. If $\phi_0(\tau)$ or $N_1(\phi(\tau))$ are polynomial function then the auxiliary functions are a sum of polynomial functions.
2. If $\phi_0(\tau)$ or $N_1(\phi(\tau))$ are exponential function then the auxiliary functions are a sum of exponential functions.
3. If $\phi_0(\tau)$ or $N_1(\phi(\tau))$ are trigonometric function then the auxiliary functions are a sum of trigonometric functions.
4. Same procedure for $\varphi(\tau)$ should be used.

Step 7: To find the numerical values of constants we used different methods, including Collocation Method, Ritz Method, Galerkin's Method, and the Least Square Method. To minimize errors, we used the Least Square Method.

3. NUMERICAL PROBLEMS:

PROBLEM NO 1:
Consider the system of Volterra integral equations in the form,

$$\begin{aligned}
&\phi(\tau)+\frac{1}{3}(\tau^4+\tau^3)-\tau^2-1-\int_0^\tau(\tau-\eta)^3\phi(\eta)\,d\eta \\
&-\int_0^\tau(\tau-\eta)^2\varphi(\eta)\,d\eta=0, \\
&\varphi(\tau)+\frac{1}{4}\left(\frac{\tau^7}{105}+\tau^5+\tau^4\right)+\tau^3-\tau-1-\int_0^\tau(\tau-\eta)^4\phi(\eta)\,d\eta \\
&-\int_0^\tau(\tau-\eta)^3\varphi(\eta)\,d\eta=0.
\end{aligned} \tag{10}$$

The exact solution of the Eq. (10) is,

$$\begin{aligned}\phi(\tau) &= -\tau^3+\tau+1,\\ \varphi(\tau) &= \tau^2+1.\end{aligned} \tag{11}$$

Source part and nonlinear part as follow,

$$\begin{cases}\phi(\tau) = -\frac{1}{3}\left(\tau^4+\tau^3\right)+\tau^2+1,\\ \varphi(\tau) = -\frac{1}{4}\left(\frac{\tau^7}{105}+\tau^5+\tau^4\right)-\tau^3+\tau+1,\\ N\left(\phi(\tau)\right) = -\int_0^\tau (\tau-\eta)^3\phi(\eta)\,d\eta-\int_0^\tau(\tau-\eta)^2\varphi(\eta)\,d\eta,\\ N\left(\varphi(\tau)\right) = -\int_0^\tau (\tau-\eta)^4\phi(\eta)\,d\eta-\int_0^\tau(\tau-\eta)^3\varphi(\eta)\,d\eta.\end{cases} \tag{12}$$

We have zero-order solution from OAFM method which are as follows,

$$\begin{aligned}\phi_0(\tau) &= -\frac{1}{3}\left(\tau^4+\tau^3\right)+\tau^2+1,\\ \varphi_0(\tau) &= -\frac{1}{4}\left(\frac{\tau^7}{105}+\tau^5+\tau^4\right)-\tau^3+\tau+1.\end{aligned} \tag{13}$$

We obtained first-order solution from the following expressions,

$$\begin{aligned}\phi_1(\tau) &= -\left(F_2 N\left(\phi_0(\tau)\right)+F_1\right),\\ \varphi_1(\tau) &= -\left(F_4 N\left(\varphi_0(\tau)\right)+F_3\right).\end{aligned} \tag{14}$$

We consider auxiliary functions for problem 1 as follows,

$$\begin{cases}F_1 = -\left(a_1\left(\frac{\tau^3}{3}+\frac{\tau^4}{3}\right)\right),\\ F_2 = 0,\\ F_3 = -\left(a_2\left(\frac{\tau^4}{4}+\frac{\tau^5}{4}+\frac{\tau^7}{420}\right)\right),\\ F_4 = 0.\end{cases} \tag{15}$$

To get first-order solutions we put Eq. (15) in Eq. (14).

$$\begin{aligned}\phi_1(\tau) &= a_1\left(\frac{\tau^3}{3}+\frac{\tau^4}{3}\right),\\ \varphi_1(\tau) &= a_2\left(\frac{\tau^4}{4}+\frac{\tau^5}{4}+\frac{\tau^7}{420}\right).\end{aligned} \tag{16}$$

By adding Eq. (13) and Eq. (16), first order OAFM solution for Eq. (10) can be obtain by the following system,

$$\begin{aligned}\tilde{\phi}(\tau) &= \phi_0(\tau)+\phi_1(\tau,a_1),\\ \tilde{\varphi}(\tau) &= \varphi_0(\tau)+\varphi_1(\tau,a_2).\end{aligned} \tag{17}$$

Here write the final solutions.

$$\phi(\tau)=1+\tau^2+\frac{1}{3}\left(-\tau^3-\tau^4\right)+a_1\left(\frac{\tau^3}{3}+\frac{\tau^4}{3}\right),$$
$$\varphi(\tau)=1+\tau-\tau^3+\frac{1}{4}\left(-\tau^4-\tau^5-\frac{\tau^7}{105}\right)+a_2\left(\frac{\tau^4}{4}+\frac{\tau^5}{4}+\frac{\tau^7}{420}\right). \tag{18}$$

Following are the residual values for $\phi(\tau)$ & $\varphi(\tau)$.

$$M\left(\phi(\tau)\right)=-\frac{\tau^3}{3}-\frac{\tau^4}{12}+\frac{\tau^6}{60}+\frac{\tau^7}{420}-a_2\frac{\tau^7}{420}+\frac{\tau^8}{672}-a_2\frac{\tau^8}{672}+\frac{\tau^{10}}{151200}$$
$$-a_2\frac{\tau^{10}}{151200}+a_1\left(\frac{\tau^3}{3}+\frac{\tau^4}{3}\right)-\frac{1}{840}\tau^4\left(210+\tau^2\left(14+\left(-1+a_1\right)\tau\left(2+\tau\right)\right)\right). \tag{19}$$

$$M\left(\varphi(\tau)\right)=-\frac{\tau^5}{5}-\frac{\tau^7}{105}+\frac{\tau^8}{840}-a_1\frac{\tau^8}{840}+\frac{\tau^9}{1890}-a_1\frac{\tau^9}{1890}$$
$$+a_2\left(\frac{\tau^4}{4}+\frac{\tau^5}{4}+\frac{\tau^7}{420}\right)-\frac{1}{554400}\tau^4\begin{pmatrix}138600+27720\tau\\ -3960\tau^3+495\left(-1+a_2\right)\tau^4\\ +275\left(-1+a_2\right)\tau^5+\left(-1+a_2\right)\tau^7\end{pmatrix}. \tag{20}$$

By applying Least Square Method, we get values of convergence control parameters, which are as follows,

$$a_1=1.`, a_2=1.`. \tag{21}$$

To get first-order approximation by OAFM for Eq. (10) by substituting Eq. (21) in Eq. (18).

$$\phi=1+\tau^2+\frac{1}{3}\left(-\tau^3-\tau^4\right)+1.`\left(\frac{\tau^3}{3}+\frac{\tau^4}{3}\right),$$
$$\varphi=1+\tau-\tau^3+\frac{1}{4}\left(-\tau^4-\tau^5-\frac{\tau^7}{105}\right)+1.`\left(\frac{\tau^4}{4}+\frac{\tau^5}{4}+\frac{\tau^7}{420}\right). \tag{22}$$

Or simplify φ we get,

$$\phi=1+\tau^2+\frac{1}{3}\left(-\tau^3-\tau^4\right)+1.`\left(\frac{\tau^3}{3}+\frac{\tau^4}{3}\right),$$
$$\varphi=1.`+\tau-1.`\tau^3. \tag{23}$$

Table 1: Comparison of the errors for problem 1.

	OAFM	**Exact**	**Biorthogonal systems**	**OAFM**
τ	$\phi(\tau)$	$-\tau^3+\tau+1$	$b_1^{\bullet}(\tau)$	$\phi^{\bullet}(\tau)$
0	1	1	0	0
0.1	1.01	1.01	3.64×10^{-5}	0
0.2	1.04	1.04	1.04×10^{-4}	0
0.3	1.09	1.09	1.75×10^{-4}	0
0.4	1.16	1.16	2.03×10^{-4}	0
0.5	1.25	1.25	1.22×10^{-4}	0
0.6	1.36	1.36	3.65×10^{-4}	0
0.7	1.49	1.49	5.91×10^{-4}	0
0.8	1.64	1.64	7.27×10^{-4}	0

0.9	1.81	1.81	6.82×10^{-4}	0
1	2	2	3.84×10^{-4}	0

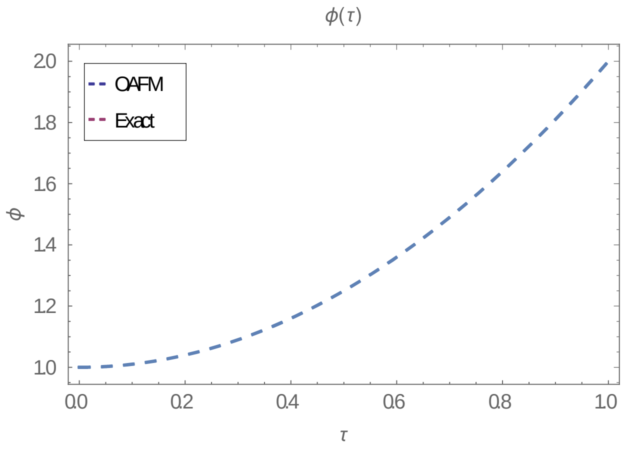

Figure 1: Plot of exact and OAFM solution for problem 1.

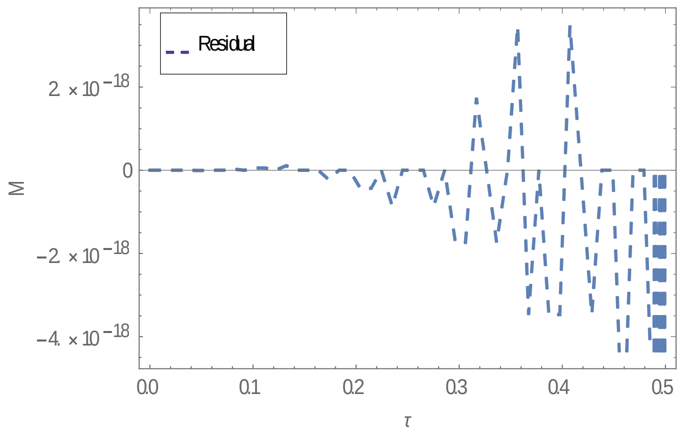

Figure 2: Plot of residual error for problem 1.

Table 2: Comparison of the errors for problem 1.

	OAFM	**Exact**	**Biorthogonal systems**	**OAFM**
τ	$\varphi(\tau)$	τ^2+1	$b_2^{\bullet}(\tau)$	$\varphi^{\bullet}(\tau)$
0	1	1	0	0
0.1	1.099	1.099	5.78×10^{-6}	2.22045×10^{-16}
0.2	1.192	1.192	3.08×10^{-5}	0
0.3	1.273	1.273	7.52×10^{-5}	2.22045×10^{-16}
0.4	1.336	1.336	1.15×10^{-4}	0
0.5	1.375	1.375	9.17×10^{-5}	0
0.6	1.384	1.384	3.15×10^{-4}	0
0.7	1.357	1.357	5.82×10^{-4}	0
0.8	1.288	1.288	8.09×10^{-4}	2.22045×10^{-16}
0.9	1.171	1.171	8.53×10^{-4}	0

1	1	1	5.01×10^{-4}	0

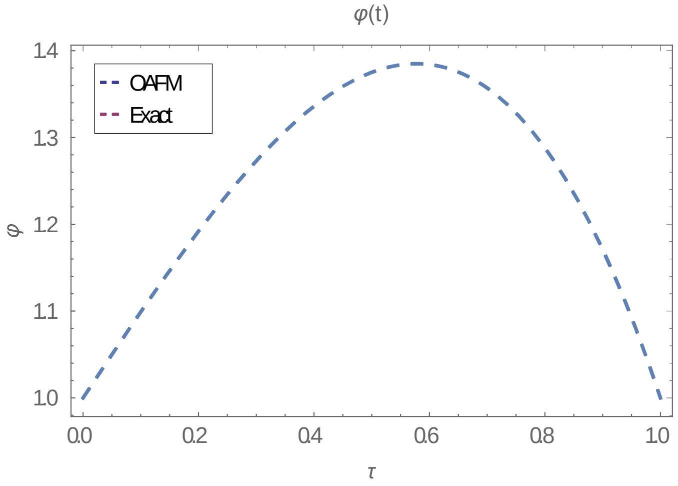

Figure 3: Plot of exact and OAFM solution for problem 1.

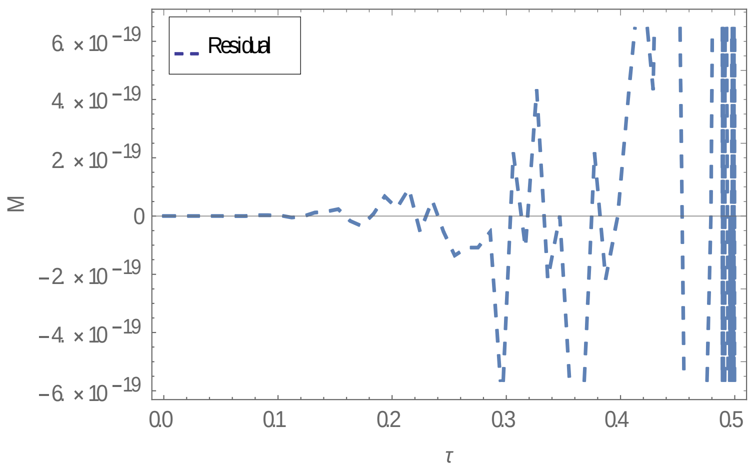

Figure 4: Plot of residual error for problem 1.

PROBLEM NO 2:

Consider the system of Volterra integral equations in the form

$$\phi(\tau)-e^{\tau}+2\tau-\int_0^{\tau}\left(e^{-\eta}\phi(\eta)+e^{\eta}\varphi(\eta)\right)d\eta=0,$$
$$\varphi(\tau)-e^{-\tau}-Sinh(2\tau)-\int_0^{\tau}\left(e^{\eta}\phi(\eta)+e^{-\eta}\varphi(\eta)\right)d\eta=0. \quad (24)$$

The exact solution of the Eq. (24) is,

$$\phi(\tau)=e^{\tau},$$
$$\varphi(\tau)=e^{-\tau}. \quad (25)$$

Source part and nonlinear part as follow,

$$\begin{cases} \phi(\tau) = e^{\tau} - 2\tau, \\ \varphi(\tau) = e^{-\tau} + Sinh(2\tau), \\ N(\phi(\tau)) = -\int_0^{\tau} \left(e^{-\eta}\phi(\eta) + e^{\eta}\varphi(\eta)\right) d\eta, \\ N(\varphi(\tau)) = -\int_0^{\tau} \left(e^{\eta}\phi(\eta) + e^{-\eta}\varphi(\eta)\right) d\eta. \end{cases} \tag{26}$$

We have zero-order solution from OAFM method which are as follows,

$$\begin{aligned} &\phi_0(\tau) = e^{\tau} - 2\tau, \\ &\varphi_0(\tau) = e^{-\tau} + Sinh(2\tau). \end{aligned} \tag{27}$$

We obtained first-order solution from the following expressions,

$$\begin{aligned} &\phi_1(\tau) = -F_1 N(\phi_0(\tau)) - F_2, \\ &\varphi_1(\tau) = -F_3 N(\varphi_0(\tau)) - F_4. \end{aligned} \tag{28}$$

We consider auxiliary functions for problem 2 as follows,

$$\begin{cases} F_1 = -\left(a_1 + 2a_2(\tau) + 2a_3(\tau^2)\right), \\ F_2 = 2a_4(\tau^3), \\ F_3 = -\left(a_5 + 2a_6(\tau^2) + 2a_7(\tau^3)\right), \\ F_4 = 2a_8(\tau^4). \end{cases} \tag{29}$$

Put Eq. (29) in Eq. (28), we obtain first-order solutions,

$$\begin{aligned} &\phi_1(\tau) = -2a_4\tau^3 - \frac{1}{6}e^{-\tau}\left(a_1 + 2a_2\tau + a_3\tau^2\right) \\ &\left(e^{4\tau} + 4e^{\tau}(-4 + 3\tau) + 3(5 + 4\tau)\right), \\ &\varphi_1(\tau) = -2a_8\tau^4 + \frac{1}{6}\left(16 - e^{-3\tau} + 3e^{-2\tau} - 3e^{\tau}\left(5 + e^{\tau} - 4\tau\right)\right) \\ &\left(a_5 + 2a_6\tau^2 + 2a_7\tau^3\right). \end{aligned} \tag{30}$$

By adding Eq. (27) and Eq. (30), first order OAFM solution for Eq. (24) can be obtain by the following system,

$$\begin{aligned} &\tilde{\phi}(\tau) = \phi_0(\tau) + \phi_1(\tau, a_1, a_2, a_3, a_4), \\ &\tilde{\varphi}(\tau) = \varphi_0(\tau) + \varphi_1(\tau, a_5, a_6, a_7, a_8). \end{aligned} \tag{31}$$

Here write the final solutions.

$$\begin{aligned} &\phi(\tau) = e^{\tau} - 2\tau - 2a_4\tau^3 - \frac{1}{6}e^{-\tau}\left(a_1 + 2a_2\tau + a_3\tau^2\right) \\ &\left(e^{4\tau} + 4e^{\tau}(-4 + 3\tau) + 3(5 + 4\tau)\right), \\ &\varphi(\tau) = e^{-\tau} - 2a_8\tau^4 + \frac{1}{6}\left(16 - e^{-3\tau} + 3e^{-2\tau} - 3e^{\tau}\left(5 + e^{\tau} - 4\tau\right)\right) \\ &\left(a_5 + 2a_6\tau^2 + 2a_7\tau^3\right) + Sinh(2\tau). \end{aligned} \tag{32}$$

Following are the residual values for $\phi(\tau)$ & $\varphi(\tau)$.

$$M\left(\phi(\tau)\right)=-2a_4\tau^3-\frac{1}{6}e^{-\tau}\left(a_1+2a_2\tau+2a_3\tau^2\right)\left(e^{-4\tau}+4e^{\tau}\left(-4+3\tau\right)+3\left(5+4\tau\right)\right)$$
$$+\frac{1}{216}\left(-27(14a_5+22a_6-39a_7\right)-4\left(-9+9a_5+4a_6-4a_7\right)+9\left(2a_5+2a_6+3a_7\right)$$
$$-108\left(-1+a_5+4a_6+12a_7\right)+576\left(a_5+4a_6-2a_7-18a_8\right)-216\tau$$
$$-144e^{\tau}\begin{pmatrix}4a_5-48a_7-72a_8+48a_7\tau+72a_8\tau-24a_7\tau^2-36a_8\tau^2+8a_7\tau^3\\+12a_8\tau^3-3a_8\tau^4+8a_6\left(2-2\tau+\tau^2\right)\end{pmatrix}$$
$$+108e^{-\tau}\left(-1+a_5+12a_7+12a_7\tau+6a_7\tau^2+2a_7\tau^3+2a_6\left(2+2\tau+\tau^2\right)\right)$$
$$+4e^{3\tau}\left(-9+9a_5-4a_7+12a_7\tau-18a_7\tau^2+18a_7\tau^3+2a_6\left(2-6\tau+9\tau^2\right)\right)$$
$$-9e^{-2\tau}\left(2a_5+a_6\left(2+4\tau+4\tau^2\right)+a_7\left(3+6\tau+6\tau^2+4\tau^3\right)\right)$$
$$-27e^{2\tau}\begin{pmatrix}2a_5\left(-7+4\tau\right)+2a_6\left(-11+22\tau-22\tau^2+8\tau^3\right)\\+a_7\left(39-78\tau+78\tau^2-52\tau^3+16\tau^4\right)\end{pmatrix}$$
$$-18\begin{pmatrix}a_1-a_2+a_3-3\left(7a_1+9a_2+11a_3\right)\\+8\left(-3+a_1-4a_2-20a_3-18a_4\right)+12\tau-e^{2\tau}\left(a_1-a_2+a_3+2a_2\tau-a_33\tau+2a_3\tau^2\right)\\+8e^{-\tau}\begin{pmatrix}3+20a_3+18a_4+3\tau+20a_3\tau+18a_4\tau+10a_3\tau^2+9a_4\tau^2+6a_3\tau^3+3a_4\tau^3\\+a_1\left(-1+3\tau\right)+a_2\left(4+4\tau+6\tau^2\right)\end{pmatrix}\\+3e^{-2\tau}\left(a_1\left(7+4\tau\right)+a_2\left(9+18\tau+8\tau^2\right)+a_3\left(11+22\tau+22\tau^2+8\tau^3\right)\right)\end{pmatrix}.$$

$$M\left(\varphi(\tau)\right)=-\frac{1}{2}+\frac{1}{54}(9-9a_5-4a_6-4a_7)+\frac{1}{384}(16a_5+4a_6+3a_7)$$
$$+\frac{1}{2}(1-a_5-4a_6+12a_7)-\frac{8}{3}(a_5+4a_6+12a_7-18a_8)+\frac{e^{-2\tau}}{2}$$
$$+2\left(1+e^{\tau}(-1+\tau)\right)+\frac{5a_5\tau}{2}-a_5\tau^2+\frac{5a_6\tau^3}{3}$$
$$-\frac{1}{4}(4a_6-5a_7)\tau^4-2a_8\tau^4-\frac{4a_7\tau^5}{5}+\frac{1}{6}\begin{pmatrix}16-e^{-3\tau}+3e^{-2\tau}\\-3e^{\tau}(5+e^{\tau}-4\tau)\end{pmatrix}$$
$$(a_5+2a_6\tau^2+2a_7\tau^3)+\frac{1}{2}e^{\tau}\begin{pmatrix}-1+a_5-12a_7+12a_7\tau-6a_7\tau^2\\+2a_7\tau^3+2a_6(2-2\tau+\tau^2)\end{pmatrix}$$
$$+\frac{2}{3}e^{-\tau}\begin{pmatrix}4a_5+48a_7-72a_8+48a_7\tau-72a_8\tau+24a_7\tau^2-36a_8\tau^2\\+8a_7\tau^3-12a_8\tau^3-3a_8\tau^4+8a_6(2+2\tau+\tau^2)\end{pmatrix}$$
$$+\frac{1}{54}e^{-3\tau}\left(-9+9a_5+4a_7+12a_7\tau+18a_7\tau^2+18a_7\tau^3+2a_6(2+6\tau+9\tau^2)\right)$$
$$-\frac{1}{384}e^{-4\tau}\left(16a_5+4a_6(1+4\tau+8\tau^2)+a_7(3+12\tau+24\tau^2+32\tau^3)\right)$$
$$+2a_4\left(6+e^{\tau}\left(-6+\tau(6+(-3+\tau)\tau)\right)\right)$$

$$+\frac{1}{96}\begin{pmatrix} -1278a_2+3327a_3 \\ +e^{4\tau}\left(4a_1+a_3+4a_3\tau(-1+2\tau)+a_2(-2+8\tau)\right) \\ +4\begin{pmatrix} 3a_1(37+4\tau(5+2\tau)) \\ +4\tau^2\left(2a_3\tau(5+3\tau)+a_2(15+8\tau)\right)\end{pmatrix}\end{pmatrix}$$
$$+64e^{\tau}\begin{pmatrix} a_1(-7+3\tau)+a_2(20-20\tau+6\tau^2) \\ +2a_3\left(-26+\tau(26+\tau(-13+3\tau))\right)\end{pmatrix}-e^{\tau}Sinh(\tau).$$

By applying Least Square Method, we get values of convergence control parameters, which are as follows,

$$\begin{aligned} &a_1=-0.9999849939218376,\ a_2=-0.03993168415808055,\\ &a_3=0.04975879985953507,\ a_4=0.4898853669189838,\\ &a_5=0.8409014926131868,\ a_6=-0.0509306757089415,\\ &a_7=0.042874534593043905,\ a_8=-0.06257219538269744.\end{aligned} \tag{33}$$

Using these values of Eq. (33) in Eq. (32), We get a first-order approximation by OAFM for the Eq. (24).

$$\phi=e^{\tau}-2\tau-0.9797707338379676`\tau^3-\frac{1}{6}e^{-\tau}$$
$$\begin{pmatrix} -0.9999849939218376` \\ -0.0798633683161611`\tau \\ +0.09951759971907014`\tau^2\end{pmatrix}\begin{pmatrix} e^{4\tau}+4e^{\tau}(-4+3\tau) \\ +3(5+4\tau)\end{pmatrix},$$

$$\varphi=e^{-\tau}+0.12514439076539488`\tau^4+\frac{1}{6}\begin{pmatrix} 16-e^{-3\tau}+3e^{-2\tau} \\ -3e^{\tau}\left(5+e^{\tau}-4\tau\right)\end{pmatrix}$$
$$\begin{pmatrix} 0.8409014926131868` \\ -0.10186135141788293`\tau^2 \\ +0.08574906918608781`\ \tau^3\end{pmatrix}+Sinh(2\tau).$$

Table 3: Comparison of the errors for problem 2.

	Trapezoidal rule solution	OAFM Solution	Exact Solution	Trapezoidal rule Abs error [20]	OAFM Abs Error
τ	$b_1(\tau)$	$\phi(\tau)$	e^{τ}	$b_1^{\bullet}(\tau)$	$\phi^{\bullet}(\tau)$
0.00	1.00000	1.00000	1.00000	0	0
0.01	1.01025	1.01007	1.01005	2×10^{-4}	1.58315×10^{-5}
0.02	1.02103	1.02027	1.02020	8.3×10^{-4}	6.45978×10^{-5}
0.03	1.03233	1.0306	1.03045	1.88×10^{-3}	1.47389×10^{-4}
0.04	1.04419	1.04108	1.04081	3.38×10^{-3}	2.65407×10^{-4}
0.05	1.05662	1.05169	1.05127	5.35×10^{-3}	4.19973×10^{-4}
0.06	1.06965	1.06245	1.06184	7.81×10^{-3}	6.12525×10^{-4}
0.07	1.08328	1.07335	1.07251	1.077×10^{-2}	8.44629×10^{-4}
0.08	1.09754	1.08441	1.08329	1.425×10^{-2}	1.11798×10^{-3}
0.09	1.11246	1.09561	1.09417	1.829×10^{-2}	1.4344×10^{-3}

0.10	1.12805	1.10697	1.10517	2.288×10^{-2}	1.79585×10^{-3}
0.11	1.14434	1.11848	1.11628	2.806×10^{-2}	2.20445×10^{-3}
0.12	1.16134	1.13016	1.12750	3.384×10^{-2}	2.66244×10^{-3}
0.13	1.17909	1.142	1.13883	4.026×10^{-2}	3.17224×10^{-3}
0.14	1.19760	1.15401	1.15027	4.733×10^{-2}	3.73639×10^{-3}
0.15	1.21690	1.16619	1.16183	2.9888×10^{-1}	4.35763×10^{-3}

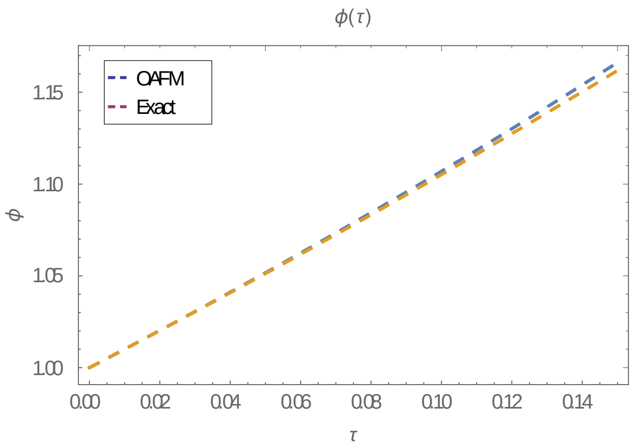

Figure 5: Plot of exact and OAFM solution for problem 2.

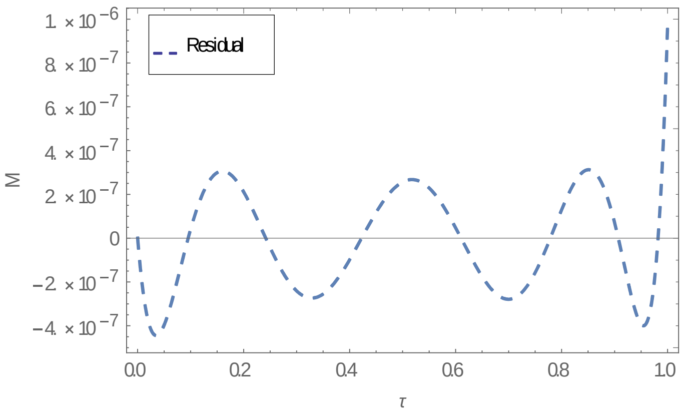

Figure 7: Plot of residual error for problem 2.

Table 4: Comparison of the errors for problem 2.

	Trapezoidal rule solution	OAFM Solution	Exact Solution	Trapezoidal rule Abs error [20]	OAFM Abs Error
τ	$b_2(\tau)$	$\varphi(\tau)$	$e^{-\tau}$	$b_2^{\bullet}(\tau)$	$\varphi^{\bullet}(\tau)$
0.00	1.00000	1.00000	1.00000	0	0
0.01	1.03025	0.993233	0.99005	4.02×10^{-2}	3.1835×10^{-3}
0.02	1.06102	0.986575	0.98020	8.082×10^{-2}	6.37619×10^{-3}
0.03	1.09232	0.980033	0.97045	1.2187×10^{-1}	9.58716×10^{-3}
0.04	1.12417	0.973615	0.96079	1.6338×10^{-1}	1.28255×10^{-2}

0.05	1.15658	0.967329	0.95123	2.0535×10^{-1}	1.61001×10^{-2}
0.06	1.18957	0.961184	0.94176	2.4781×10^{-1}	1.94199×10^{-2}
0.07	1.22316	0.955188	0.93239	2.9077×10^{-1}	2.27938×10^{-2}
0.08	1.25737	0.949347	0.92312	3.3425×10^{-1}	2.62306×10^{-2}
0.09	1.29221	0.94367	0.91393	3.7828×10^{-1}	2.9739×10^{-2}
0.10	1.32771	0.938165	0.90484	4.2287×10^{-1}	3.33278×10^{-2}
0.11	1.36389	0.93284	0.89583	4.6806×10^{-1}	3.70056×10^{-2}
0.12	1.40076	0.927702	0.88692	5.1384×10^{-1}	4.07811×10^{-2}
0.13	1.43835	0.922758	0.87810	5.6025×10^{-1}	4.46628×10^{-2}
0.14	1.47669	0.918018	0.86936	6.0733×10^{-1}	4.86594×10^{-2}
0.15	1.51578	0.913487	0.86071	6.5507×10^{-1}	5.27795×10^{-2}

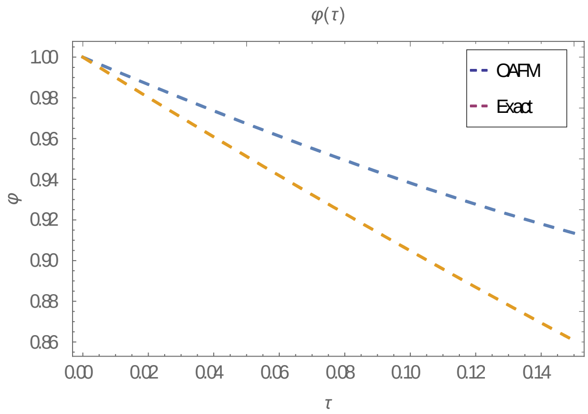

Figure 7: Plot of exact and OAFM solution for problem 2.

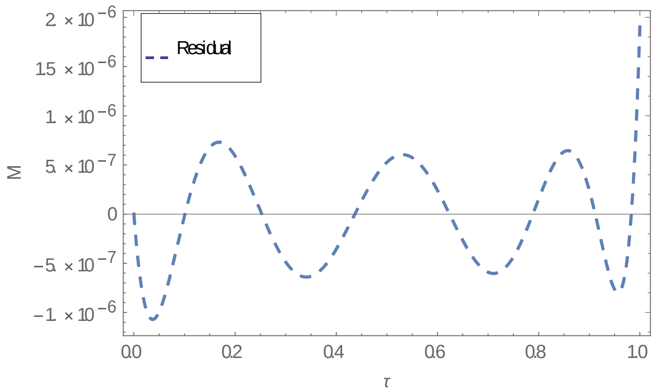

Figure 8: Plot of residual error for problem 2.

PROBLEM NO 3:

Consider the system of Volterra integral equations in the form,

$$\begin{aligned}&\phi(\tau)-\left(-3\,Cos(\tau)+3\right)-\int_0^\tau Sin(\tau-\eta)\phi(\eta)\,d\eta+\int_0^\tau Cos(\tau-\eta)\varphi(\eta)\,d\eta=0,\\&\varphi(\tau)-\left(e^{\tau}+\frac{\tau^3}{3}-1\right)+\int_0^\tau\phi(\eta)\,d\eta+\int_0^\tau e^{\tau-\eta}\varphi(\eta)\,d\eta=0.\end{aligned}\tag{34}$$

The exact solution of the Eq. (34) is,

$$\phi(\tau)=\tau^2,\quad \varphi(\tau)=\tau. \tag{35}$$

Source part and nonlinear part as follow,

$$\begin{cases}\phi(\tau)=-3\,Cos(\tau)+3,\\ \varphi(\tau)=e^{\tau}+\dfrac{\tau^3}{3}-1,\\ N(\phi(\tau))=-\int_0^{\tau}Sin(\tau-\eta)\phi(\eta)\,d\eta+\int_0^{\tau}Cos(\tau-\eta)\varphi(\eta)\,d\eta,\\ N(\varphi(\tau))=\int_0^{\tau}\phi(\eta)\,d\eta+\int_0^{\tau}e^{\tau-\eta}\varphi(\eta)\,d\eta.\end{cases} \tag{36}$$

We have zero-order solution from OAFM method which are as follows,

$$\phi_0(\tau)=-3\,Cos(\tau)+3,\quad \varphi_0(\tau)=e^{\tau}+\frac{\tau^3}{3}-1. \tag{37}$$

We obtained first-order solution from the following expressions,

$$\phi_1(\tau)=-\left(F_1N(\phi_0(\tau))+F_2\right),\quad \varphi_1(\tau)=-\left(F_4N(\varphi_0(\tau))+F_3\right). \tag{38}$$

We consider auxiliary functions for problem 3 as follows,

$$\begin{cases}F_1=0,\\ F_2=-\left(a_1\left(-\tau^2-3\,Cos(\tau)+3\right)\right),\\ F_3=-\left(a_2\left(\tau-e^{\tau}-\dfrac{\tau^3}{3}+1\right)\right),\\ F_4=0.\end{cases} \tag{39}$$

Using these values of Eq. (39) in Eq. (38), we get first-order solutions,

$$\phi_1(\tau)=a_1\left(-\tau^2-3\,Cos(\tau)+3\right),\quad \varphi_1(\tau)=a_2\left(\tau-e^{\tau}-\frac{\tau^3}{3}+1\right). \tag{40}$$

By adding Eq. (37) and Eq. (40), first order OAFM solution for Eq. (34) can be obtain by the following system,

$$\tilde{\phi}(\tau)=\phi_0(\tau)+\phi_1(\tau,a_1),\quad \tilde{\varphi}(\tau)=\varphi_0(\tau)+\varphi_1(\tau,a_2). \tag{41}$$

Here write the final solutions.

$$\phi(\tau)=3+a_1\left(-\tau^2-3\,Cos(\tau)+3\right)-3\,Cos(\tau),\quad \varphi(\tau)=e^{\tau}+\frac{\tau^3}{3}-1+a_2\left(\tau-e^{\tau}-\frac{\tau^3}{3}+1\right). \tag{42}$$

Following are the residual values for $\phi(\tau)$ & $\varphi(\tau)$.

$$M\left(\phi(\tau)\right)=-3-5a_1+a_1\tau^2+a_1\left(3-\tau^2-3Cos(\tau)\right)+\left(3+5a_1\right)Cos(\tau)$$
$$+\frac{3}{2}(1+a_1)\tau Sin(\tau)+\frac{1}{2}\begin{pmatrix}-4+6a_2+e^{-\tau}-a_2e^{-\tau}+2\tau^2-2a_2\tau^2\\+(3-5a_2)Cos(\tau)+(-1+a_2)Sin(\tau)\end{pmatrix}.$$

$$M\left(\varphi(\tau)\right)=-1+3(1+a_1)\tau+(-2+a_2)\tau+(-1+a_2)\tau^2-\frac{a_1\tau^3}{3}+\frac{1}{3}(-1+a_2)\tau^3$$
$$+e^{\tau}\left(1+\tau-a_2\tau\right)+a_2\left(1-e^{\tau}+\tau-\frac{\tau^3}{3}\right)-3(1+a_1)Sin(\tau).$$

By applying Least Square Method, we get values of convergence control parameters, which are as follows,

$$a_1=-1.`,a_2=1.`. \tag{43}$$

To get a first-order approximation by OAFM for the Eq. (34) putting these values of Eq. (43) in Eq. (42).

$$\phi=3-1.`\left(-\tau^2-3\,Cos(\tau)+3\right)-3\,Cos(\tau),$$
$$\varphi=-1+e^{\tau}+\frac{\tau^3}{3}+1.`\left(\tau-e^{\tau}-\frac{\tau^3}{3}+1\right).$$

Or simplify the above equation we get,

$$\phi=0.`+1.`\tau^2,$$
$$\varphi=0.`+1.`\tau.$$

Table 5: Comparison of the errors for problem 3.

	OAFM Solution	Exact Solution	RMCM Abs Error [21]	OAFM Abs Error
τ	$\phi(\tau)$	τ^2	$AE(\varphi_1)$	$\phi^{\bullet}(\tau)$
0.0	0	0	0	0
0.1	0.01	0.01	8.20×10^{-6}	2.15106×10^{-16}
0.2	0.04	0.04	1.34×10^{-5}	2.77556×10^{-17}
0.3	0.09	0.09	8.52×10^{-5}	1.66533×10^{-16}
0.4	0.16	0.16	1.73×10^{-4}	1.11022×10^{-16}
0.5	0.25	0.25	1.37×10^{-4}	0
0.6	0.36	0.36	3.70×10^{-4}	2.22045×10^{-16}
0.7	0.49	0.49	5.89×10^{-3}	1.11022×10^{-16}
0.8	0.64	0.64	1.51×10^{-4}	0
0.9	0.81	0.81	6.75×10^{-4}	0
1.0	1	1	3.018×10^{-3}	0

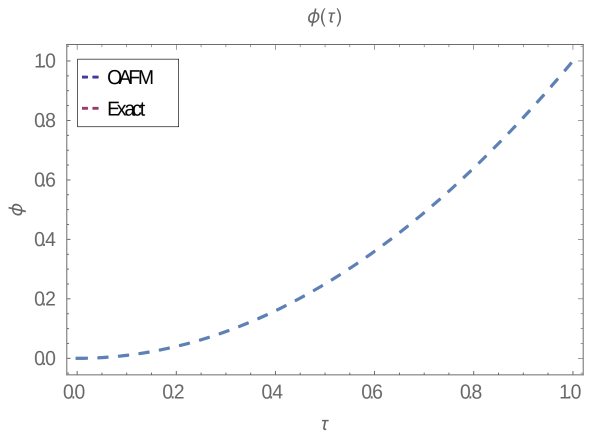

Figure 9: Plot of exact and OAFM solution for problem 3.

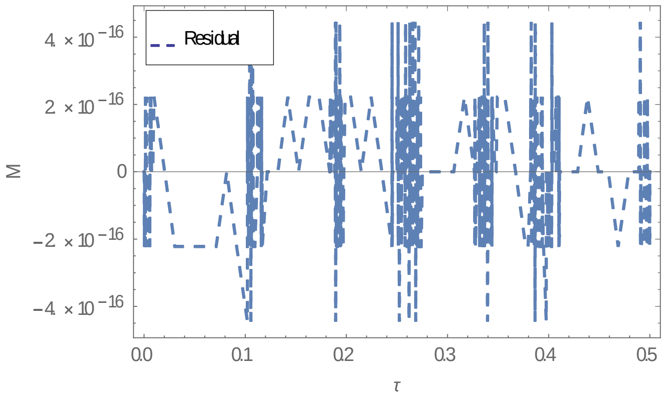

Figure 10: Plot of residual error for problem 3.

Table 6: Comparison of the errors for problem 3.

	OAFM Solution	Exact Solution	RMCM Abs Error [21]	OAFM Abs Error
τ	$\varphi(\tau)$	τ	$AE(\varphi_2)$	$\varphi^{\bullet}(\tau)$
0.0	0	0	0	0
0.1	0.1	0.1	6.93×10^{-7}	0
0.2	0.2	0.2	7.58×10^{-6}	0
0.3	0.3	0.3	8.10×10^{-5}	0
0.4	0.4	0.4	5.79×10^{-5}	0
0.5	0.5	0.5	7.86×10^{-5}	0
0.6	0.6	0.6	2.57×10^{-4}	0
0.7	0.7	0.7	9.308×10^{-3}	1.11022×10^{-16}
0.8	0.8	0.8	5.47×10^{-4}	0
0.9	0.9	0.9	8.75×10^{-5}	0
1.0	1.0	1.0	3.943×10^{-3}	0

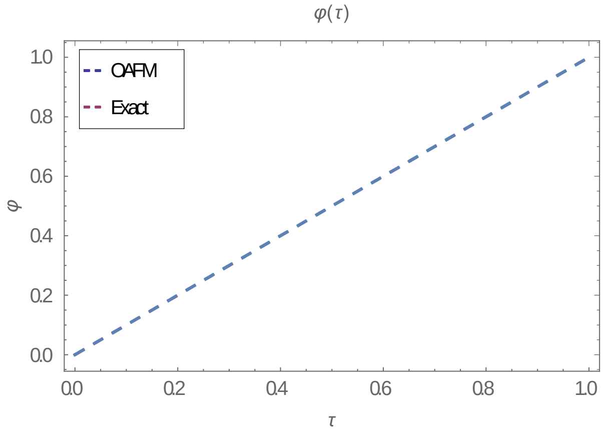

Figure 11: Plot of exact and OAFM solution for problem 3.

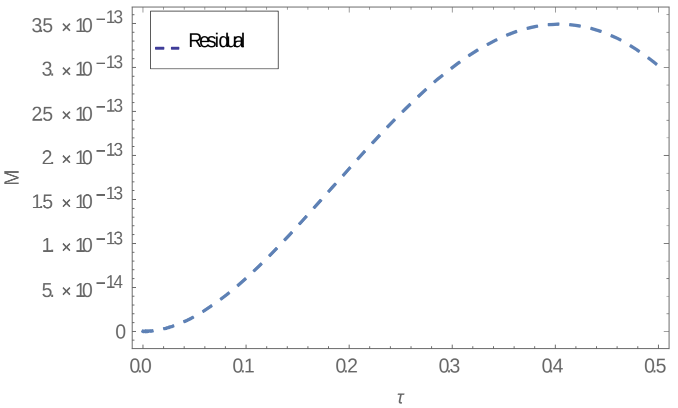

Figure 12: Plot of residual error for problem 3.

4. Conclusion

In this work, we applied an Optimal Auxiliary Function Method for solving Volterra integral equations systems of the second kind. In particular, the OAFM consists of the auxiliary functions and some parameters that guarantee a very rapid convergence of solution. According to our study, the obtained approximation after only one iteration is considered very good. The results obtained by OAFM for system of integral equations are compared with Biorthogonal systems in a Banach space Fixed point, the implicit Trapezoidal rule in conjunction with Newton's method, RMCM in the form of tables and figures (See Fig. 1, Fig. 3, Fig. 5, Fig. 7, Fig. 9, Fig. 11). Residual error of problems are shown in (See Fig. 2, Fig. 4, Fig. 6, Fig. 8, Fig. 10, Fig. 12) According to the numerical results which obtaining from the illustrative examples, we conclude that our method is more powerful for the solution convergence than the other methods and can be applied for more physical like [36, 37, 38, 39] in future.

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